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One-dimensional impenetrable anyons in thermal equilibrium: I. Anyonic generalization of Lenard's formula

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Abstract

We have obtained an expansion of the reduced density matrices (or, equivalently, correlation functions of the fields) of impenetrable one-dimensional anyons in terms of the reduced density matrices of fermions using the mapping between anyon and fermion wavefunctions. This is the generalization to anyonic statistics of the result obtained by Lenard (1966 *J. Math. Phys.* **7** 1268) for bosons. In the case of impenetrable but otherwise free anyons with statistics parameter κ, the anyonic reduced density matrices in the grand canonical ensemble are expressed as Fredholm minors of the integral operator $(1 - \gamma\hat{\theta}_T)$ with the complex statistics-dependent coefficient $\gamma = (1 + e^{\pm i\pi\kappa})/\pi$. For $\kappa = 0$ we recover the bosonic case of Lenard $\gamma = 2/\pi$. Due to nonconservation of parity, the anyonic field correlators $\langle\Psi_A^\dagger(x')\Psi_A(x)\rangle$ are different depending on the sign of $x' - x$.

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1. Introduction

During the last few years, several models of one-dimensional anyons [2–6] have attracted considerable attention. This attention is motivated, besides general fundamental physics interest of anyons, by several new statistical-mechanical features of the anyonic systems. One of them is the behavior of the zero-temperature field correlation functions, which exhibit oscillations with the period dependent on the statistics parameter in the leading term of their

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algebraically decaying large-distance asymptotics. These oscillations were obtained within the harmonic fluid approach [7] and conformal field theory [8].

Much of this effort was focused on the anyonic model proposed in [4] and clarified in [6, 8]. This model can be understood as an anyonic extension of the Bose gas with δ-function interaction [9] which was solved by Lieb and Liniger. The study of this model was initiated by Batchelor *et al* [11–13] and continued in [7, 8, 14]. In the limit of infinite interaction strength, the anyons become impenetrable. The present work is the first paper in a series that intends to provide a complete treatment of the correlation functions of such impenetrable free anyons in a manner similar to the impenetrable Bose gas [15]. The first step taken in this work is the anyonic generalization of Lenard's formula [1] which gives the reduced density matrices of hard-core bosons in terms of the fermionic reduced density matrices. With this result, the reduced density matrices of free impenetrable bosons can be expressed in terms of Fredholm minors of the integral operator $\hat{\theta}_T$ with the kernel given by the Fourier transform of the Fermi distribution function.

In this work, we use the Anyon–Fermi [5, 6] and Anyon–Bose [4] mapping to generalize the treatment of Lenard to the case of anyons. Our main result is theorem 4.1 which in the case of impenetrable free anyons leads to a representation of the correlation functions of anyonic fields as Fredholm minors of the operator $\hat{\theta}_T$. This representation of the correlation functions can be summarized as follows. The Fermi distribution function $\vartheta(k, T, h)$ at temperature T and chemical potential h defines the integral operator $\hat{\theta}_T$ with the kernel

$$\theta_T(x-y) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}k(x-y)}\vartheta(k, T, h), \qquad \vartheta(k, T, h) = \frac{1}{1+\mathrm{e}^{(k^2-h)/T}}, \tag{1}$$

which acts on an arbitrary function $f(x)$ as

$$(\hat{\theta}_T f)(x) = \int_I \theta_T(x-y) f(y)\,\mathrm{d}y, \tag{2}$$

where I is an interval or a finite union of intervals on the real axis. The resolvent kernel $\varrho_T(x, y)$ associated with the kernel $\theta_T(x, y)$ is defined to satisfy the equation

$$\varrho_T(x, y) - \gamma \int_I \theta_T(x-z)\varrho_T(z, y)\,\mathrm{d}z = \theta_T(x-y). \tag{3}$$

Then, in the thermodynamic limit, the temperature-dependent correlator of the anyonic field operators is given by the following relations:

- If $x' > x$,

$$\left\langle \Psi_A^{\dagger}(x')\Psi_A(x)\right\rangle_{T,h} = \frac{1}{\pi}\varrho_T(x', x)\det(1-\gamma\hat{\theta}_T)|_{\gamma=(1+\mathrm{e}^{+\mathrm{i}\pi\kappa})/\pi}, \tag{4}$$

  where $\det(1-\gamma\hat{\theta}_T)$ is the Fredholm determinant of the integral operator $\hat{\theta}_T$ (see appendix A) which acts on the interval $I_+ = [x, x']$.
- If $x' < x$,

$$\left\langle \Psi_A^{\dagger}(x')\Psi_A(x)\right\rangle_{T,h} = \frac{1}{\pi}\varrho_T(x, x')\det(1-\gamma\hat{\theta}_T)|_{\gamma=(1+\mathrm{e}^{-\mathrm{i}\pi\kappa})/\pi}, \tag{5}$$

  and now the integral operator acts on the interval $I_- = [x', x]$.

Using equation (A.11) we can also express these formulae in terms of the first Fredholm minor of the integral operator $\hat{\theta}_T$.

The methods used in our work allows for a direct extension of these results to the $2n$-point correlators with certain ordering of the arguments:

- If $x_1 < x_1' < \cdots < x_n < x_n'$,

  $$\left\langle \Psi_A^\dagger(x_n') \cdots \Psi_A^\dagger(x_1') \Psi_A(x_1) \cdots \Psi_A(x_n) \right\rangle_{T,h}$$
  $$= \frac{C(x_1', \ldots, x_n)}{\pi^n} \varrho_T \begin{pmatrix} x_1, \ldots, x_n \\ x_1', \ldots, x_n' \end{pmatrix} \det(1 - \gamma \hat{\theta}_T)|_{\gamma=(1+\mathrm{e}^{+\mathrm{i}\pi\kappa})/\pi}, \tag{6}$$

  where now the integral operator acts in the region $I_+ = [x_1, x_1'] \cup \cdots \cup [x_n, x_n']$, the statistics factor C is defined by equations (51), (22) and (23), and other notations are given in appendix A.
- If $x_1' < x_1 \cdots < x_n' < x_n$,

  $$\left\langle \Psi_A^\dagger(x_n') \cdots \Psi_A^\dagger(x_1') \Psi_A(x_1) \cdots \Psi_A(x_n) \right\rangle_{T,h}$$
  $$= \frac{C(x_1', \ldots, x_n)}{\pi^n} \varrho_T \begin{pmatrix} x_1, \ldots, x_n \\ x_1', \ldots, x_n' \end{pmatrix} \det(1 - \gamma \hat{\theta}_T)|_{\gamma=(1+\mathrm{e}^{-\mathrm{i}\pi\kappa})/\pi}, \tag{7}$$

  where the integral operator acts in the region $I_- = [x_1', x_1] \cup \cdots \cup [x_n', x_n]$.

Again, equation (A.11) can be used to express the $2n$-point correlators as the nth Fredholm minors of the integral operator $\hat{\theta}_T$.

The paper is organized as follows. In section 2, we introduce the reduced density matrices of anyons and their relation to the correlation functions of the anyonic fields. In section 3, we describe the mappings between the wavefunctions of the one-dimensional particles with Fermi, Bose and Anyonic statistics. In section 4, we prove the main theorem which expresses the reduced density of anyons in terms of the reduced density matrices of bosons and fermions. The consequences of this theorem in the case of free impenetrable particles is analyzed in section 5, where we obtain the reduced density matrices of anyons as Fredholm minors of an integral operator. The basic information on Fredholm determinants, details of calculations with anyonic fields, and a sketch of the derivation of the reduced density matrices for free fermions in the grand canonical ensemble are presented in three appendices.

2. From correlation functions to reduced density matrices

The one-dimensional anyons considered in this work are characterized by anyonic fields $\Psi_A^\dagger(x)$, $\Psi_A(x)$ which obey the following commutation relations:

$$\Psi_A(x_1)\Psi_A^\dagger(x_2) = \mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(x_1-x_2)}\Psi_A^\dagger(x_2)\Psi_A(x_1) + \delta(x_1 - x_2), \tag{8}$$
$$\Psi_A^\dagger(x_1)\Psi_A^\dagger(x_2) = \mathrm{e}^{\mathrm{i}\pi\kappa\epsilon(x_1-x_2)}\Psi_A^\dagger(x_2)\Psi_A^\dagger(x_1), \tag{9}$$
$$\Psi_A(x_1)\Psi_A(x_2) = \mathrm{e}^{\mathrm{i}\pi\kappa\epsilon(x_1-x_2)}\Psi_A(x_2)\Psi_A(x_1). \tag{10}$$

Here κ is the statistics parameter, and $\epsilon(x) = x/|x|, \epsilon(0) = 0$. The commutation relations become bosonic for $\kappa = 0$ and fermionic for $\kappa = 1$. For an arbitrary Hamiltonian of the anyons confined to the interval $V = [-L/2, L/2]$, the N-particle eigenstates are defined as

$$|\Psi_N(\{\lambda\})\rangle = \frac{1}{\sqrt{N!}} \int_V \mathrm{d}z_1 \cdots \int_V \mathrm{d}z_N \, \chi_N^a(z_1, \ldots, z_N|\{\lambda\}) \Psi_A^\dagger(z_N) \cdots \Psi_A^\dagger(z_1)|0\rangle, \tag{11}$$

$$\langle\Psi_N(\{\lambda\})| = \frac{1}{\sqrt{N!}} \int_V \mathrm{d}z_1 \cdots \int_V \mathrm{d}z_N \, \langle 0|\Psi_A(z_1) \cdots \Psi_A(z_N) \chi_N^{*a}(z_1, \ldots, z_N|\{\lambda\}), \tag{12}$$

where χ_N^a are the (norm-one) quantum-mechanical wavefunctions of N anyons, and $\{\lambda\}$ is a set of quantum numbers specifying the state. Note the order of the field operators in these relations, which is dictated by the boundary conditions on the wavefunctions in the periodic or quasi-periodic case [6]. The wavefunctions χ have the anyonic symmetry

$$\chi_N^a(z_1,\ldots,z_i,z_{i+1},\ldots,z_N)=\mathrm{e}^{\mathrm{i}\pi\kappa\epsilon(z_i-z_{i+1})}\chi_N^a(z_1,\ldots,z_{i+1},z_i,\cdots,z_N), \quad (13)$$

that reflects the field commutation relations. We are interested in computing the finite-temperature correlation functions of anyonic fields. The simplest example of these correlators is

$$\left\langle\Psi_A^\dagger(x')\Psi_A(x)\right\rangle. \quad (14)$$

In the grand canonical ensemble characterized by temperature T and chemical potential h, the field correlation function is given by following relation:

$$\left\langle\Psi_A^\dagger(x')\Psi_A(x)\right\rangle_{T,h}=\sum_{N=1}^{\infty}\sum_{\{\lambda\}}\mathrm{e}^{hN/T}\frac{\mathrm{e}^{-E(\{\lambda\})/T}}{Z(h,V,T)}\langle\Psi_N(\{\lambda\})|\Psi_A^\dagger(x')\Psi_A(x)|\Psi_N(\{\lambda\})\rangle, \quad (15)$$

where $E(\{\lambda\})$ is the energy of the eigenstate with quantum numbers $\{\lambda\}$, and $Z(h,V,T)$ is the grand-canonical partition function

$$Z(h,V,T)=\sum_{N=0}^{\infty}\sum_{\{\lambda\}}\mathrm{e}^{hN/T}\,\mathrm{e}^{-E(\{\lambda\})/T}. \quad (16)$$

As shown in appendix B, the correlator at a fixed number of particles N is given by an overlap integral of the corresponding wavefunction

$$\langle\Psi_N(\{\lambda\})|\Psi_A^\dagger(x')\Psi_A(x)|\Psi_N(\{\lambda\})\rangle=N\int_V \mathrm{d}z_1\cdots$$
$$\times\int_V \mathrm{d}z_{N-1}\chi_N^{*a}(z_1,\ldots,z_{N-1},x'|\{\lambda\})\chi_N^a(z_1,\ldots,z_{N-1},x|\{\lambda\}), \quad (17)$$

so that the correlation function (15) can be written as

$$\left\langle\Psi_A^\dagger(x')\Psi_A(x)\right\rangle_{T,h}=\sum_{N=1}^{\infty}\sum_{\{\lambda\}}\mathrm{e}^{hN/T}\frac{\mathrm{e}^{-E(\{\lambda\})/T}}{Z(h,V,T)}N\int_V \mathrm{d}z_1\cdots\int_V \mathrm{d}z_{N-1}$$
$$\times\chi_N^{*a}(z_1,\ldots,z_{N-1},x'|\{\lambda\})\chi_N^a(z_1,\ldots,z_{N-1},x|\{\lambda\}). \quad (18)$$

We will also be interested in a class of $2n$-point field correlation functions at finite temperature,

$$\left\langle\Psi_A^\dagger(x_n')\cdots\Psi_A^\dagger(x_1')\Psi_A(x_1)\cdots\Psi_A(x_n)\right\rangle_{T,h}. \quad (19)$$

These correlators can be expressed similarly to equation (18)

$$\left\langle\Psi_A^\dagger(x_n')\cdots\Psi_A^\dagger(x_1')\Psi_A(x_1)\cdots\Psi_A(x_n)\right\rangle_{T,h}$$
$$=\sum_{N=n}^{\infty}\sum_{\{\lambda\}}\mathrm{e}^{hN/T}\frac{\mathrm{e}^{-E(\{\lambda\})/T}}{Z(h,V,T)}\frac{N!}{(N-n)!}\int_V \mathrm{d}z_1\cdots\int_V \mathrm{d}z_{N-n}$$
$$\times\chi_N^{*a}(z_1,\ldots,z_{N-n},x_1',\ldots,x_n'|\{\lambda\})\chi_N^a(z_1,\ldots,z_{N-n},x_1,\ldots,x_n|\{\lambda\}). \quad (20)$$

As one can see from (18) and (20), the correlation functions are obtained as a combination of the wavefunctions and ensemble probabilities. In this context, it is useful, similarly to the case of fermionic or bosonic particles, to introduce the reduced density matrices of anyons:

Definition 1. *For a statistical ensemble characterized by the probabilities $p^N_{\{\lambda\}}$, the anyonic n-particle reduced density matrix is defined as*

$$\left(x_1,\ldots,x_n\middle|\rho_n^a\middle|x_1',\ldots,x_n'\right)=\sum_{N=n}^{\infty}\sum_{\{\lambda\}}p^N_{\{\lambda\}}\frac{N!}{(N-n)!}\int_V \mathrm{d}z_1\cdots\int_V\mathrm{d}z_{N-n}$$
$$\times\,\chi_N^{*a}(z_1,\ldots,z_{N-n},x_1',\ldots,x_n'|\{\lambda\})\chi_N^a(z_1,\ldots,z_{N-n},x_1,\ldots,x_n|\{\lambda\}),\qquad(21)$$

where the wavefunctions $\chi^a_{N,\{\lambda\}}$ are normalized to one.

If the probabilities $p^N_{\{\lambda\}}$ coincide with those in the grand-canonical ensemble, $p^N_{\{\lambda\}}=\mathrm{e}^{Nh/T}\mathrm{e}^{-E(\{\lambda\})/T}/Z(h,V,T)$, the one-particle reduced density matrix is just the 2-point correlator (18), and the n-particle reduced density matrix is the particular $2n$-point correlator (20). These relations are exactly the same as in the case of bosonic and fermionic statistics. A particular 'anyonic' feature of definition 1 is the fact that we integrate over the first $N-n$ arguments of the wavefunctions. In the case of bosonic and fermionic reduced density matrices, integration over any subset of the $N-n$ out of N arguments produces the same result due to the parity of the wavefunctions. This is not the case for the reduced density matrices of anyons due to the anyonic symmetry (13) which in general, e.g., in the periodic or quasi-periodic situation ('anyons on a ring') makes different arguments of the wavefunctions inequivalent (see the discussion in the following section).

Under a certain set of conditions, which is also made precise in the following section, there is a correspondence between the anyonic and the fermionic or bosonic wavefunctions. This correspondence will be used later to express the anyonic reduced density matrices as expansions in terms of fermionic or bosonic ones.

3. Anyon–Fermi and Anyon–Bose mapping

To establish the correspondence between the wavefunctions of anyons and fermions or bosons we define the two functions which essentially incorporate statistical properties of the wavefunctions of different statistics in one dimension,

$$A_\kappa(z_1,\ldots,z_N)=\mathrm{e}^{\mathrm{i}\pi\kappa\sum_{j<k}\epsilon(z_j-z_k)/2}\qquad(22)$$

and

$$B(z_1,\ldots,z_N)=\prod_{j>k}\epsilon(z_j-z_k),\qquad(23)$$

where the notations are the same as in equation (8).

The mapping between anyons and fermions or bosons is analogous to the Bose–Fermi mapping discovered in [10], where it was noted that any wavefunction of N fermions has a bosonic counterpart given by

$$\chi^b(z_1,\ldots,z_N)=B(z_1,\ldots,z_N)\chi^f(z_1,\ldots,z_N).\qquad(24)$$

This correspondence is valid under very general conditions, with no restrictions on the external or particle–particle interaction potential, except for the requirement of the hard-core condition which should make the bosons impenetrable. For particles confined to a box with 'hard-wall' boundary conditions (BC), bosonic and fermionic wavefunctions satisfy the same BC. In this case, if χ^f is an eigenfunction of the Hamiltonian, then χ^b is also an eigenfunction with the same eigenvalue. However, in the case of a ring of length L with periodic BC for bosons, the BC for the fermions are in general different and given by

$$\chi^f(0,\ldots,z_N)=(-1)^{N-1}\chi^f(L,\ldots,z_N).\qquad(25)$$

For even N, when equation (25) means that the BC for fermions and bosons are different by a phase shift π, the relation between the eigenenergies of the fermionic and bosonic systems is less direct. Since for non-coincident coordinates $B^2 \equiv 1$, the Bose–Fermi mapping (24) is symmetric and remains true if the superscripts b and f are interchanged.

3.1. Anyon–Fermi mapping

It is straightforward to see that similarly to the Bose–Fermi mapping (24), the wavefunction with anyonic symmetry (13) can be obtained by multiplication of a fermionic wavefunction with the statistics factors (22) and (23) [5, 6],

$$\chi^a(z_1,\ldots,z_N) = A_\kappa(z_1,\ldots,z_N)B(z_1,\ldots,z_N)\chi^f(z_1,\ldots,z_N), \tag{26}$$

Besides the anyonic symmetry (13), the wavefunction χ^a (26) satisfies the condition

$$\chi^a(z_1,\ldots,z_N)|_{z_i=z_j} = 0 \qquad \text{for all} \quad \{i,j\}\in\{1,\ldots,N\}. \tag{27}$$

This means that the correspondence (26) is valid as long as potential energy contains a hard-core part which ensures that the anyons are impenetrable and condition (27) is indeed satisfied. Other properties of the Anyon–Fermi mapping (26) are similar to those of the Bose–Fermi mapping. It is valid for an arbitrary form of the potential energy in the particle Hamiltonian. When the particles are confined to an interval with *'hard-wall' boundary conditions*, the fermionic and anyonic systems both satisfy the same BC of the wavefunctions vanishing at the ends of the interval. In this case, if χ^f is an eigenfunction of the particle Hamiltonian, then χ^a is also an eigenfunction of this Hamiltonian with the same eigenvalue. This follows from the fact that the statistics factors (22) and (23) are constant everywhere except for the points of coincident coordinates, where the wavefunctions vanish.

When particles are confined to a ring with *periodic or quasi-periodic BC*, the properties of the Anyon–Fermi mapping are more complicated. In this case, the anyonic wavefunction will have different boundary conditions for each of its coordinate (see [6] and appendix A of [8]), the difference being given by an extra phase shift that depends on the statistics parameter κ. Specifically, if the fermion wavefunction obeys some generic quasi-periodic BC (the same in all coordinates) which can be written as

$$\chi^f(0,\ldots,z_N) = (-1)^{N-1}\,\mathrm{e}^{-\mathrm{i}\phi}\chi^f(L,\ldots,z_N), \tag{28}$$

then the anyonic wavefunction obeys the following BC in its different arguments:

$$\begin{aligned}
&\chi^a(0,z_2,\ldots,z_N) = \mathrm{e}^{-\mathrm{i}\overline{\phi}}\chi^a(L,z_2,\ldots,z_N),\\
&\chi^a(z_1,0,\ldots,z_N) = \mathrm{e}^{\mathrm{i}(2\pi\kappa-\overline{\phi})}\chi^a(z_1,L,\ldots,z_N),\\
&\vdots\\
&\chi^a(z_1,z_2,\ldots,0) = \mathrm{e}^{\mathrm{i}(2(N-1)\pi\kappa-\overline{\phi})}\chi^a(z_1,z_2\cdots,L),
\end{aligned} \tag{29}$$

where $\overline{\phi} = \phi+\pi\kappa(N-1)$. As for the Bose–Fermi mapping (24) with even N, the anyonic and fermionic eigenenergies are not related directly in the situation of a ring with quasiperiodic BC. In physics terms, this difference between anyons and fermions corresponds to the statistical magnetic flux $\pi\kappa(N-1)$ through the ring produced by N one-dimensional anyons of statistics κ.

The Anyon–Fermi mapping (26) is not symmetric. The inverse relation can be written as

$$\chi^f(z_1,\ldots,z_N) = A_{-\kappa}(z_1,\ldots,z_N)B(z_1,\ldots,z_N)\chi^a(z_1,\ldots,z_N). \tag{30}$$

3.2. Anyon–Bose mappping

The Anyon–Bose mapping was historically the first mapping of this kind introduced for one-dimensional anyons in [4],

$$\chi^a(z_1,\ldots,z_N) = A_\kappa(z_1,\ldots,z_N)\chi^b(z_1,\ldots,z_N). \tag{31}$$

The wavefunction χ^a in (31) has the correct anyonic symmetry (13), and in contrast to Anyon–Fermi mapping (26), need not vanish when a pair of coordinates coincide. It should be noted, however, that without this hard-core condition, the discontinuity of the statistics factor A_κ (22) at coinciding coordinates translates into discontinuity of the wavefunctions (31). In this case one needs an additional condition regularizing the wavefunctions. Also, without the hard-core condition, the statistics factor A changes substantially the behavior of the wavefunctions at the points of coincident coordinates (for instance, if the particle–particle interaction is δ-functional, statistics renormalizes the interaction strength [8]) and the energy eigenvalues of the bosonic and anyonic problems are different regardless of the boundary conditions.

Other properties of the Anyon–Bose mapping (31) are very similar to those of the Anyon–Fermi mapping. It is valid for arbitrary potential energy. For particles in a box with 'hard-wall' BC, both wavefunctions (31) satisfy the same condition $\chi^a = 0$ and $\chi^b = 0$ at the boundary. For particles on a ring with generic quasi-periodic BC for bosons that can be written as

$$\chi^b(0,\ldots,z_N) = \mathrm{e}^{-\mathrm{i}\phi}\chi^b(L,\ldots,z_N) \tag{32}$$

(and have the same form for all other arguments of χ^b), the anyonic wavefunction obeys the following BC:

$$\begin{aligned}
&\chi^a(0,z_2,\ldots,z_N) = \mathrm{e}^{-\mathrm{i}\overline{\phi}}\chi^a(L,z_2,\ldots,z_N),\\
&\chi^a(z_1,0,\ldots,z_N) = \mathrm{e}^{\mathrm{i}(2\pi\kappa-\overline{\phi})}\chi^a(z_1,L,\ldots,z_N),\\
&\vdots\\
&\chi^a(z_1,z_2,\ldots,0) = \mathrm{e}^{\mathrm{i}(2(N-1)\pi\kappa-\overline{\phi})}\chi^a(z_1,z_2\cdots,L),
\end{aligned} \tag{33}$$

where $\overline{\phi} = \phi + \pi\kappa(N-1)$. The Anyon–Bose mapping is also not symmetric. The inverse of (31) is

$$\chi^b(z_1,\ldots,z_N) = A_{-\kappa}(z_1,\ldots,z_N)\chi^a(z_1,\ldots,z_N). \tag{34}$$

4. Main theorem

In this section, we consider an arbitrary statistical ensemble in which the states $\chi^a_{N,\{\lambda\}}$ occur with probabilities $p^N_{\{\lambda\}}$. The anyons are assumed to be confined to an interval $V = [-L/2, L/2]$, and wavefunctions are normalized to 1: $\|\chi^a_{N,\{\lambda\}}\| = 1$. Our goal is to establish a relation between the reduced density matrices ρ^a_n of anyons (21) and similarly defined reduced density matrices of bosons ρ^b_m and fermions ρ^f_p. The fermionic and bosonic states that correspond to the anyonic states $\chi^a_{N,\{\lambda\}}$ in the Anyon–Fermi (26) or Anyon–Bose (31) mappings have similarly normalized wavefunctions, and we assume that they have the same probabilities $p^N_{\{\lambda\}}$. This assumption is natural under the conditions (discussed in the previous section) for which the energies of the states of different statistics are the same, as they are, for instance, when the wavefunctions satisfy the 'hard-wall' boundary conditions and hard-core condition on the particle–particle interaction. The relation between the reduced density matrices is established by the following theorem:

Theorem 4.1. *Let $x_1,\ldots,x_n,x_1',\ldots,x_n'$ be $2n$ coordinates in the interval V, and $O_\pm$ are the parts of the space of these coordinates in which they are ordered, respectively, as $x_1 < x_1' < \cdots < x_n < x_n'$ and $x_1' < x_1 < \cdots < x_n' < x_n$. For the O_+ ordering, one can define the subset of V: $I_+ = [x_1,x_1']\cup\cdots\cup[x_n,x_n']\subset V$, and the subset $I_- = [x_1',x_1]\cup\cdots\cup[x_n',x_n]\subset V$ for ordering as in O_-. If the conditions of validity of the Anyon–Fermi (26) or the Anyon–Bose mapping (31) are fulfilled, the reduced density matrices of anyons can be expressed then in terms of the reduced density matrices of fermions as*

$$\left(x_1,\ldots,x_n\middle|\rho_n^a\middle|x_1',\ldots,x_n'\right)_\pm = A_{-\kappa}(x_1',\cdots,x_n')B(x_1',\ldots,x_n')A_\kappa(x_1,\ldots,x_n)B(x_1,\ldots,x_n)$$
$$\times\sum_{j=0}^{\infty}(-1)^j\frac{(1+\mathrm{e}^{\pm\mathrm{i}\pi\kappa})^j}{j!}\int_{I_\pm}\mathrm{d}z_1\cdots$$
$$\times\int_{I_\pm}\mathrm{d}z_j\,(x_1,\ldots,x_n,z_1,\ldots,z_j|\rho_{n+j}^f|x_1',\ldots,x_n',z_1,\cdots,z_j) \tag{35}$$

or bosons as

$$\left(x_1,\ldots,x_n\middle|\rho_n^a\middle|x_1',\ldots,x_n'\right)_\pm = A_{-\kappa}(x_1',\ldots,x_n')A_\kappa(x_1,\ldots,x_n)\sum_{j=0}^{\infty}(-1)^j\frac{(1-\mathrm{e}^{\pm\mathrm{i}\pi\kappa})^j}{j!}$$
$$\times\int_{I_\pm}\mathrm{d}z_1\cdots\int_{I_\pm}\mathrm{d}z_j\left(x_1,\ldots,x_n,z_1,\ldots,z_j\middle|\rho_{n+j}^b\middle|x_1',\ldots,x_n',z_1,\cdots,z_j\right). \tag{36}$$

The subscript $\pm$ in these expressions specifies whether $x_1,\ldots,x_n,x_1',\ldots,x_n'$ are ordered as in O_+ or O_-.

Proof. The proof follows that of Lenard [1], generalizing it to the anyonic statistics. As a first step, we need a preliminary result.

Lemma 4.1. *For any symmetric function $f(z_1,\ldots,z_n)$ and a constant α,*

$$\int_V\mathrm{d}z_1\cdots\int_V\mathrm{d}z_n\alpha^{\sigma(I_\pm)}f(z_1,\ldots,z_n)$$
$$=\sum_{j=0}^{n}C_j^n(-1+\alpha)^j\int_{I_\pm}\mathrm{d}z_1\cdots\int_{I_\pm}\mathrm{d}z_j\int_V\mathrm{d}z_{j+1}\cdots\int_V\mathrm{d}z_n\,f(z_1,\ldots,z_n), \tag{37}$$

where $C_j^n = \frac{n!}{(n-j)!j!}$ and $\sigma(I_\pm)$ is the number of variables $z_1,\ldots,z_n$ contained in $I_\pm$.

Proof. The LHS of (37) can be written explicitly as

$$Q=\sum_{m=0}^{n}C_m^n\alpha^m\int_{I_\pm}\mathrm{d}z_1\cdots\int_{I_\pm}\mathrm{d}z_m\int_{V\setminus I_\pm}\mathrm{d}z_{m+1}\cdots\int_{V\setminus I_\pm}\mathrm{d}z_n\,f(z_1,\ldots,z_n), \tag{38}$$

and combined with an obvious relation $\int_{V\setminus I_\pm}\mathrm{d}z_i=\int_V\mathrm{d}z_i-\int_{I_\pm}\mathrm{d}z_i$, $(i=m+1,\ldots,n)$ can be further transformed into

$$Q=\sum_{m=0}^{n}C_m^n\alpha^m\sum_{k=0}^{n-m}C_k^{n-m}(-1)^k\int_{I_\pm}\mathrm{d}z_1\cdots\int_{I_\pm}\mathrm{d}z_{m+k}\int_V\mathrm{d}z_{m+k+1}\cdots\int_V\mathrm{d}z_n\,f(z_1,\ldots,z_n). \tag{39}$$

Collecting the terms in this expression with the same $j=m+k$, we obtain the desired result

$$Q=\sum_{j=0}^{n}C_j^n(-1+\alpha)^j\int_{I_\pm}\mathrm{d}z_1\cdots\int_{I_\pm}\mathrm{d}z_j\int_V\mathrm{d}z_{j+1}\cdots\int_V\mathrm{d}z_n\,f(z_1,\cdots,z_n).$$

□

Now we can prove theorem 4.1 starting with (35). Using the Anyon–Fermi mapping (26) we have

$$\begin{aligned}\big(x_1,\ldots,x_n\big|\rho_n^a\big|x_1',\ldots,x_n'\big)_\pm &= \sum_{N=n}^{\infty}\sum_{\{\lambda\}} p_{\{\lambda\}}^N \frac{N!}{(N-n)!}\int_V \mathrm{d}z_1\cdots\int_V \mathrm{d}z_{N-n} C(x_1,\ldots,x_n,x_1',\ldots,x_n')_\pm \\ &\quad\times \chi_N^{*f}(z_1,\ldots,z_{N-n},x_1',\ldots,x_n'|\{\lambda\})\chi_N^f(z_1,\ldots,z_{N-n},x_1,\ldots,x_n|\{\lambda\}),\end{aligned} \tag{40}$$

where

$$\begin{aligned}C(x_1,\ldots,x_n,x_1',\ldots,x_n')_\pm &= A_{-\kappa}(x_1',\cdots,x_n')B(x_1',\ldots,x_n')A_\kappa(x_1,\ldots,x_n)B(x_1,\ldots,x_n) \\ &\quad\times\prod_{j=1}^{n}\prod_{i=1}^{N-n}\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(z_i-x_j')/2}\mathrm{e}^{\mathrm{i}\pi\kappa\epsilon(z_i-x_j)/2}\epsilon(x_j'-z_i)\epsilon(x_j-z_i).\end{aligned} \tag{41}$$

One can see directly that

$$\prod_{j=1}^{n}\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(z-x_j')/2}\,\mathrm{e}^{\mathrm{i}\pi\kappa\epsilon(z-x_j)/2}\epsilon(x_j'-z)\epsilon(x_j-z) = \begin{cases}-\mathrm{e}^{\pm\mathrm{i}\pi\kappa}, & z \quad \text{in } I_\pm,\\ 1, & z \quad \text{not in } I_\pm.\end{cases} \tag{42}$$

This means that

$$\begin{aligned}C(x_1,\ldots,x_n,x_1',\ldots,x_n')_\pm &= A_{-\kappa}(x_1',\ldots,x_n') \\ &\quad\times B(x_1',\ldots,x_n')A_\kappa(x_1,\ldots,x_n)B(x_1,\ldots,x_n)(-\mathrm{e}^{\pm\mathrm{i}\pi\kappa})^{\sigma'(I_\pm)},\end{aligned} \tag{43}$$

where $\sigma'(I_\pm)$ is the number of variables $z_1,\ldots,z_{N-n}$ in $I_\pm$. Applying now lemma 4.1 with $\alpha = -\mathrm{e}^{\pm\mathrm{i}\pi\kappa}$, we obtain for the anyonic reduced density matrices

$$\begin{aligned}\big(x_1,\ldots,x_n\big|\rho_n^a\big|x_1',\ldots,x_n'\big)_\pm &= A_{-\kappa}(x_1',\ldots,x_n')B(x_1',\ldots,x_n') \\ &\quad\times A_\kappa(x_1,\ldots,x_n)B(x_1,\ldots,x_n)\sum_{N=n}^{\infty}\sum_{\{\lambda\}}p_{\{\lambda\}}^N\frac{N!}{(N-n)!}\sum_{j=0}^{N-n}C_j^{N-n}(-1)^j \\ &\quad\times(1+\mathrm{e}^{\pm\mathrm{i}\pi\kappa})^j\int_{I_\pm}\mathrm{d}z_1\cdots\int_{I_\pm}\mathrm{d}z_j\int_V\mathrm{d}z_{j+1}\cdots\int_V\mathrm{d}z_{N-n}\chi_{N,\{\lambda\}}^{*f}\chi_{N,\{\lambda\}}^f.\end{aligned} \tag{44}$$

Interchanging the order of summations, one can note that the sum over N and $\{\lambda\}$ is precisely ρ_{n+j}^f. Therefore finally

$$\begin{aligned}\big(x_1,\ldots,x_n\big|\rho_n^a\big|x_1',\ldots,x_n\big)_\pm &= A_{-\kappa}(x_1',\cdots,x_n')B(x_1',\ldots,x_n') \\ &\quad\times A_\kappa(x_1,\ldots,x_n)B(x_1,\ldots,x_n)\sum_{j=0}^{\infty}(-1)^j\frac{(1+\mathrm{e}^{\pm\mathrm{i}\pi\kappa})^j}{j!}\int_{I_\pm}\mathrm{d}z_1\cdots \\ &\quad\times\int_{I_\pm}\mathrm{d}z_j\big(x_1,\ldots,x_n,z_1,\cdots,z_j\big|\rho_{n+j}^f\big|x_1',\ldots,x_n',z_1,\ldots,z_j\big).\end{aligned} \tag{45}$$

The proof of (36) is similar. In this case, we use the Anyon–Bose mapping (31), and the $C_\pm$ function is

$$\begin{aligned}C(x_1,\ldots,x_n,x_1',\ldots,x_n')_\pm &= A_{-\kappa}(x_1',\cdots,x_n')A_\kappa(x_1,\ldots,x_n) \\ &\quad\times\prod_{j=1}^{n}\prod_{i=1}^{N-n}\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(z_i-x_j')/2}\,\mathrm{e}^{\mathrm{i}\pi\kappa\epsilon(z_i-x_j)/2}.\end{aligned} \tag{46}$$

This means that we can use lemma 4.1 with $\alpha = e^{\pm i\pi\kappa}$. Interchanging the order of summation and identifying the bosonic reduced density matrices ρ^b_{n+j} we obtain (36). □

The results of theorem 4.1 do not depend on the statistical ensemble used in the computation of the reduced density matrices as long as the particles are subject to the hard-wall boundary conditions making the state energies independent of the statistics. They also do not depend on the form of the interparticle potential beyond the need for the hard-core part which ensures that the wavefunctions satisfy the hard-core condition. If both of these conditions are satisfied, we can see from (35) and (36) that there is also no explicit dependence on the length L of the confining box V, and the results remain valid in the thermodynamic limit $L \to \infty$.

5. Impenetrable free case

The Anyon–Fermi relation derived above for the reduced density matrices is particularly useful in the situation when the radius of the hard-core interaction is vanishingly small, and no other interactions are present. In this case, the fermionic problem is identical to free fermions, since the hard-core potential of zero radius effectively vanished due to antisymmetry of the wavefunctions. The reduced density matrices ρ^f_n coincide then with those of free fermions [1] (see appendix C):

$$\left(x_1,\ldots,x_n\middle|\rho^f_n\middle|x'_1,\ldots,x'_n\right) = \frac{1}{\pi^n}\theta_T\begin{pmatrix}x_1,\ldots,x_n\\ x'_1,\ldots,x'_n\end{pmatrix}, \tag{47}$$

where $\theta_T(x,y)/\pi$ is the Fourier transform of the Fermi distribution function

$$\theta_T(x,y) = \frac{1}{2}\int_{-\infty}^{\infty} dk\,\frac{e^{ik(x-y)}}{1+e^{(k^2-h)/T}}. \tag{48}$$

At $T = 0$ we have

$$\theta_0(x,y) = \frac{\sin q(x-y)}{x-y}, \tag{49}$$

where $q = \sqrt{h}$ is the Fermi momentum.

Applying theorem 4.1 and (C.8) we have

$$\begin{aligned}\left(x_1,\ldots,x_n\middle|\rho^a_n\middle|x'_1,\ldots,x'_n\right)_\pm &= C\sum_{j=0}^{\infty}(-1)^j\frac{(1+e^{\pm i\pi\kappa})^j}{j!}\int_{I_\pm}dz_1\cdots\\ &\quad\times\int_{I_\pm}dz_j\,\frac{1}{\pi^{n+j}}\theta_T\begin{pmatrix}x_1,\ldots,x_n, & z_1,\ldots,z_j\\ x'_1,\ldots,x'_n, & z_1,\ldots,z_j\end{pmatrix},\end{aligned} \tag{50}$$

where

$$C(x'_1,\ldots,x_n) \equiv A_{-\kappa}(x'_1,\cdots,x'_n)B(x'_1,\ldots,x'_n)A_\kappa(x_1,\ldots,x_n)B(x_1,\ldots,x_n) \tag{51}$$

and the subscript $\pm$ specifies particular ordering of $x_1,\ldots,x_n,x'_1,\ldots,x'_n$ as in theorem 4.1. This result can be rewritten in terms of Fredholm minors using (A.10)

$$\begin{aligned}&\left(x_1,\ldots,x_n\middle|\rho^a_n\middle|x'_1,\ldots,x'_n\right)_\pm\\ &\qquad= \frac{1}{\pi^n}C(x'_1,\ldots,x_n)\det\left(1-\gamma\hat\theta^\pm_T\,\middle|\,\begin{matrix}x_1,\ldots,x_n\\ x'_1,\ldots,x'_n\end{matrix}\right)\Bigg|_{\gamma=(1+e^{\pm i\pi\kappa})/\pi},\end{aligned} \tag{52}$$

where the integral operator $\hat\theta^\pm_T$ with kernel $\theta_T(x,y)$ is defined by its action on an arbitrary function f,

$$\left(\hat\theta^\pm_T f\right)(x) = \int_{I_\pm}\theta_T(x,y)f(y)\,dy. \tag{53}$$

Finally, introducing the resolvent kernel $\varrho_T^{\pm}(x, y)$ associated with the kernel $\theta_T(x, y)$, which satisfies

$$\varrho_T^{\pm}(x, y) - \frac{(1 + \mathrm{e}^{\pm i\pi\kappa})}{\pi} \int_{I_\pm} \theta_T(x - z)\varrho_T^{\pm}(z, y)\, \mathrm{d}z = \theta_T(x - y), \tag{54}$$

and making use of (A.11), (20) and (18), we obtain

$$\begin{aligned}&\left\langle \Psi_A^\dagger(x_n') \cdots \Psi_A^\dagger(x_1')\Psi_A(x_1) \cdots \Psi_A(x_n)\right\rangle_{T,h,\pm} \\ &\quad = \frac{C(x_1', \ldots, x_n)}{\pi^n} \varrho_T^{\pm}\begin{pmatrix} x_1, \ldots, x_n \\ x_1', \ldots, x_n' \end{pmatrix} \det\left(1 - \gamma\hat{\theta}_T^{\pm}\right)\Big|_{\gamma=(1+\mathrm{e}^{\pm i\pi\kappa})/\pi}.\end{aligned} \tag{55}$$

In the particular case of the simplest two-point correlator, this expression reduces to

$$\langle\Psi_A^\dagger(x')\Psi_A(x)\rangle_{T,h,\pm} = \frac{1}{\pi}\varrho_T^{\pm}(x', x) \det\left(1 - \gamma\hat{\theta}_T^{\pm}\right)\Big|_{\gamma=(1+\mathrm{e}^{\pm i\pi\kappa})/\pi}, \tag{56}$$

and gives the correlator of two anyonic fields in terms of the Fredholm determinant of the integral operator $\hat{\theta}_T$ and its resolvent kernel.

6. Conclusions

In this work, we have studied the field correlation functions (reduced density matrices) for free impenetrable anyons in one dimension and obtained the representation for these functions in terms of the Fredholm minors of the integral operator with the kernel given by (48). This representation of the correlation functions generalizes similar results [1] for one-dimensional impenetrable bosons, and can be used to study the asymptotic behavior of the correlation functions beyond the approximation of conformal invariance. In the case of bosons, there is also an alternative representation of the zero-temperature one-particle reduced density matrix as the determinant of a Toeplitz matrix, development of which was motivated by the study of momentum distribution of bosons in the ground state [10, 16, 17]. This representation was extended recently to anyons by Santachiara *et al* [14] in their study of the entanglement entropy of impenetrable free anyons. As in the case of bosons, explicit demonstration of the equivalence of the two representations for anyonic correlation functions is an open problem.

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Appendix A. Fredholm determinants

In this appendix, we give a brief summary of results of Fredholm theory of integral equations. For more details, see, e.g., [18]. Consider the Fredholm equation of the second kind

$$f(x) - \gamma \int_a^b K(x, y) f(y)\, \mathrm{d}y = g(x), \tag{A.1}$$

where the kernel $K(x, y)$ is a symmetric, bounded and continuous function.

Defining operations with kernel $K(x, y)$ similarly to the usual matrix operations,

$$K^n(x, y) = \int_a^b K(x, z)K^{n-1}(z, y)\, \mathrm{d}z, \qquad \text{with} \qquad K^1(x, y) = K(x, y), \tag{A.2}$$

and

$$\mathrm{Tr}\, K = \int_a^b K(x, x)\, \mathrm{d}x, \qquad \mathrm{Tr}\, K^2 = \int_a^b \int_a^b K(x, y)K(y, x)\, \mathrm{d}x\, \mathrm{d}y \quad \text{and so on,} \tag{A.3}$$

we have the formulae that are useful for calculation of the Fredholm determinant of the integral operator $1-\gamma\hat{K}$,

$$(1-\gamma\hat{K})^{-1} = 1+\gamma K^1+\gamma^2 K^2+\cdots, \tag{A.4}$$

and

$$\ln\det(1-\gamma\hat{K}) = -\sum_{n=1}^{\infty}\frac{\gamma^n}{n}\mathrm{Tr}\,K^n. \tag{A.5}$$

Indeed, writing (A.5) as

$$\det(1-\gamma\hat{K}) = \prod_{n=1}^{\infty}\exp\left\{-\frac{\gamma^n}{n}\mathrm{Tr}\,K^n\right\}, \tag{A.6}$$

and collecting terms of the same order in γ one can see that the determinant can be written conveniently as

$$\det(1-\gamma\hat{K}) = \sum_{n=0}^{\infty}(-1)^n\frac{\gamma^n}{n!}\int_a^b \mathrm{d}x_1\cdots\int_a^b \mathrm{d}x_n\, K_n\begin{pmatrix}x_1,\ldots,x_n\\ x_1,\ldots,x_n\end{pmatrix}, \tag{A.7}$$

where

$$K_n\begin{pmatrix}x_1,\ldots,x_n\\ y_1,\ldots,y_n\end{pmatrix} \equiv \det_{1\leqslant j,k\leqslant n}[K(x_j,y_k)]. \tag{A.8}$$

The resolvent kernel $R(x,y)$ associated with the kernel $K(x,y)$ is defined as $\hat{R} = (1-\gamma\hat{K})^{-1}\hat{K}$, i.e.,

$$R(x,y)-\gamma\int_a^b K(x,z)R(z,y)\,\mathrm{d}z = K(x,y). \tag{A.9}$$

If one introduces the determinants R_n of kernels R similarly to (A.8), an important relation can be proven to exist between R_n and the rth Fredholm minor defined as a natural generalization of equation (A.7),

$$\det\left(1-\gamma\hat{K}\left|\begin{matrix}y_1, & \cdots, & y_r\\ y_1', & \cdots, & y_r'\end{matrix}\right.\right) = \sum_{n=0}^{\infty}(-1)^n\frac{\gamma^n}{n!}$$
$$\times\int_a^b \mathrm{d}x_1\cdots\int_a^b \mathrm{d}x_n\,K_{n+r}\begin{pmatrix}y_1, & \cdots, & y_r, & x_1, & \cdots, & x_n\\ y_1', & \cdots, & y_r', & x_1, & \cdots, & x_n\end{pmatrix}. \tag{A.10}$$

The relation is [19]

$$\det\left(1-\gamma\hat{K}\left|\begin{matrix}y_1, & \cdots, & y_r\\ y_1', & \cdots, & y_r'\end{matrix}\right.\right) = \det\left(1-\gamma\hat{K}\right)R_n\begin{pmatrix}y_1,\ldots,y_r\\ y_1',\ldots,y_r'\end{pmatrix}. \tag{A.11}$$

Appendix B. Anyonic correlators

In this appendix, we prove equation (17) following the approach used in [6] for calculation of the anyonic matrix elements. We start with the simple case of the correlator

$$\langle\Psi_2|\Psi_A^\dagger(x')\Psi_A(x)|\Psi_2\rangle, \tag{B.1}$$

where we have omitted the quantum numbers $\{\lambda\}$ unimportant for the present computation. From (11) and (12), we have

$$\langle\Psi_2|\Psi_A^\dagger(x')\Psi_A(x)|\Psi_2\rangle = \frac{1}{2}\int \mathrm{d}z^2\,\mathrm{d}y^2\,\chi_2^{*a}(y_1,y_2)\chi_2^a(z_1,z_2)\langle 0|\Psi_A(y_1)$$
$$\times\Psi_A(y_2)\Psi_A^\dagger(x')\Psi_A(x)\Psi_A^\dagger(z_2)\Psi_A^\dagger(z_1)|0\rangle. \tag{B.2}$$

Defining for the moment

$$A = \langle 0|\Psi_A(y_1)\Psi_A(y_2)\Psi_A^\dagger(x')\Psi_A(x)\Psi_A^\dagger(z_2)\Psi_A^\dagger(z_1)|0\rangle, \tag{B.3}$$

and using the commutation relation (8), $\Psi_A(x)|0\rangle = 0$, and $\langle 0|0\rangle = 1$ we obtain

$$\begin{aligned} A &= \langle 0|\Psi_A(y_1)\Psi_A(y_2)\Psi_A^\dagger(x')\big[\Psi_A^\dagger(z_2)\Psi_A(x)\,\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(x-z_2)} + \delta(x-z_2)\big]\Psi_A^\dagger(z_1)|0\rangle,\\ &= \langle 0|\Psi_A(y_1)\Psi_A(y_2)\Psi_A^\dagger(x')\Psi_A^\dagger(z_2)\Psi_A(x)\Psi_A^\dagger(z_1)|0\rangle\,\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(x-z_2)}\\ &\quad+\langle 0|\Psi_A(y_1)\Psi_A(y_2)\Psi_A^\dagger(x')\Psi_A^\dagger(z_1)|0\rangle\delta(x-z_2),\\ &= \underbrace{\langle 0|\Psi_A(y_1)\Psi_A(y_2)\Psi_A^\dagger(x')\Psi_A^\dagger(z_2)|0\rangle}_{(\mathbf{a})}\,\delta(x-z_1)\,\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(x-z_2)}\\ &\quad+\underbrace{\langle 0|\Psi_A(y_1)\Psi_A(y_2)\Psi_A^\dagger(x')\Psi_A^\dagger(z_1)|0\rangle}_{(\mathbf{b})}\,\delta(x-z_2). \end{aligned} \tag{B.4}$$

Performing similar transformations we find that

$$\begin{aligned} \mathbf{a} &= \delta(y_1-x')\delta(y_2-z_2)\,\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(y_2-x')} + \delta(y_2-x')\delta(y_1-z_2),\\ \mathbf{b} &= \delta(y_1-x')\delta(y_2-z_1)\,\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(y_2-x')} + \delta(y_1-z_1)\delta(y_2-x'). \end{aligned} \tag{B.5}$$

Substitution of (B.4) and (B.5) into (B.2) gives

$$\begin{aligned} &\langle\Psi_2|\Psi_A^\dagger(x')\Psi_A(x)|\Psi_2\rangle\\ &\quad= \frac{1}{2}\int \mathrm{d}z_1\big\{\chi_2^{*a}(x',z_1)\chi_2^a(x,z_1)\mathrm{e}^{-\mathrm{i}\pi\kappa[\epsilon(z_1-x')+\epsilon(x-z_1)]} + \chi_2^{*a}(z_1,x')\chi_2^a(x,z_1)\\ &\qquad\times\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(x-z_1)}\chi_2^{*a}(x',z_1)\chi_2^a(z_1,x)\,\mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(z_1-x')} + \chi_2^{*a}(z_1,x')\chi_2^a(z_1,x)\big\}. \end{aligned} \tag{B.6}$$

Anyonic property of the wavefunctions (13) together with its complex conjugate

$$\chi_N^{*a}(z_1,\ldots,z_i,z_{i+1},\ldots,z_N) = \mathrm{e}^{-\mathrm{i}\pi\kappa\epsilon(z_i-z_{i+1})}\chi_N^{*a}(z_1,\ldots,z_{i+1},z_i,\cdots,z_N) \tag{B.7}$$

means that (B.6) reduces to a simple form

$$\langle\Psi_2|\Psi_A^\dagger(x')\Psi_A(x)|\Psi_2\rangle = 2\int \mathrm{d}z_1\,\chi_2^{*a}(z_1,x')\chi_2^a(z_1,x). \tag{B.8}$$

The generalization to the N-particle eigenstate is straightforward and gives

$$\langle\Psi_N|\Psi_A^\dagger(x')\Psi_A(x)|\Psi_N\rangle = N\int \mathrm{d}z^{N-1}\,\chi_N^{*a}(z_1,\ldots,z_{N-1},x')\chi_N^a(z_1,\ldots,z_{N-1},x). \tag{B.9}$$

Appendix C. Reduced density matrices of free fermions

The reduced density matrices for free 1D fermions were calculated in the original paper of Lenard [1]. To make our discussion self-contained, we provide here a sketch of the proof and the main results in the notations that in general allow for an external potential $U(z)$ acting on the particles.

We assume that the fermions are confined to the domain $V = [-L/2, L/2]$, and have a complete set $\{u_\lambda(z)\}$ of normalized single-particle wavefunctions with energies ϵ_λ in the potential $U(z)$. For instance, for $U(z)\equiv 0$, and the 'hard-wall' boundary conditions at the boundaries of the domain V,

$$u_\lambda(z) = \begin{cases} \sqrt{\frac{2}{L}}\sin(\lambda z),\ \lambda = \frac{2\pi}{L}, \frac{4\pi}{L}, \ldots,\\ \sqrt{\frac{2}{L}}\cos(\lambda z),\ \lambda = \frac{\pi}{L}, \frac{3\pi}{L}, \ldots, \end{cases} \tag{C.1}$$

and, with appropriate conventions, $\epsilon_\lambda = \lambda^2$. The N-body wavefunction of a stationary state is given by the Slater determinant

$$\chi_N^f(z_1, \ldots, z_N|\{\lambda\}) = \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} (-1)^\pi \prod_{i=1}^{N} u_{\lambda_i}(z_{\pi(i)}), \tag{C.2}$$

where the set $\{\lambda\}$ consists of non-coincident single-particle states λ_i, and the energy eigenvalue is $E(\{\lambda\}) = \sum_{i=1}^{N} \epsilon_{\lambda_i}$. In the grand-canonical ensemble, the Gibbs measure is

$$p_{\{\lambda\}}^N = \mathrm{e}^{hN/T} \frac{\mathrm{e}^{-E(\{\lambda\})/T}}{Z(h, L, T)}, \qquad \text{with} \qquad Z(h, L, T) = \sum_{N=0}^{\infty} \sum_{\{\lambda\}} \mathrm{e}^{hN/T} \mathrm{e}^{-E(\{\lambda\})/T}, \tag{C.3}$$

where h is the chemical potential. Using the fact that, with an extra factor $1/N!$ included to compensate for overcounting, summation over $\{\lambda\}$ can be replaced with summation over independent individual λ_i's, one obtains the following fundamental formula:

$$\sum_{\{\lambda\}} \mathrm{e}^{hN/T} \mathrm{e}^{-E(\{\lambda\})/T} \chi_N^{*f}(z_1, \ldots, z_N|\{\lambda\}) \chi_N^f(z_1', \ldots, z_N'|\{\lambda\})$$

$$= \frac{1}{N!} \sum_{\pi \in S_N} (-1)^\pi \prod_{i=1}^{N} F(z_i, z_{\pi(i)}') \tag{C.4}$$

$$= \frac{1}{N!} F_N \begin{pmatrix} z_1, \ldots, z_N \\ z_1', \ldots, z_N' \end{pmatrix}. \tag{C.5}$$

Here we have used (A.8) and

$$F(x, y) \equiv \mathrm{e}^{h/T} \sum_{\lambda} \mathrm{e}^{-\epsilon_\lambda/T} u_\lambda^*(x) u_\lambda(y). \tag{C.6}$$

Equation (C.4) and proper normalization of the wavefunctions u_λ show that the fermionic statistical sum (C.3) can be expressed as the determinant (A.7) of the integral operator with kernel (C.6),

$$Z(h, L, T) = \sum_{N=0}^{\infty} \frac{1}{N!} \int_V \mathrm{d}z_1 \cdots \int_V \mathrm{d}z_N F_N \begin{pmatrix} z_1, \ldots, z_N \\ z_1, \ldots, z_N \end{pmatrix} = \det(1 + \hat{F}). \tag{C.7}$$

Similarly, using the definition (A.10) of Fredholm minor of the same operator we see that

$$\sum_{N=n}^{\infty} \mathrm{e}^{hN/T} \sum_{\{\lambda\}} \mathrm{e}^{-E(\{\lambda\})/T} \frac{N!}{(N-n)!} \int_V \mathrm{d}z_1 \cdots \int_V \mathrm{d}z_{N-n}$$

$$\times \chi_N^{*f}(z_1, \ldots, z_{N-n}, x_1, \ldots, x_n|\{\lambda\}) \chi_N^f(z_1, \ldots, z_{N-n}, x_1', \ldots, x_n'|\{\lambda\})$$

$$= \det \left(1 + \hat{F} \left| \begin{matrix} x_1, \ldots, x_n \\ x_1', \ldots, x_n' \end{matrix} \right. \right), \tag{C.8}$$

so that the reduced density matrix of the fermions can be expressed as

$$\left(x_1, \ldots, x_n \middle| \rho_n^f \middle| x_1', \ldots, x_n'\right) = \det \left(1 + \hat{F} \left| \begin{matrix} x_1, \ldots, x_n \\ x_1', \ldots, x_n' \end{matrix} \right. \right) \Big/ \det(1 + \hat{F}). \tag{C.9}$$

The relation (A.11) for Fredholm minors means that this result can be expressed simply in terms of the resolvent kernel $\theta_T(x, y)/\pi$ associated with kernel $F(x, y)$ (C.6) (factors of π are chosen so that the notations are the same as in the main text),

$$\left(x_1, \ldots, x_n \middle| \rho_n^f \middle| x_1', \ldots, x_n'\right) = \frac{1}{\pi^n} \theta_T \begin{pmatrix} x_1, \ldots, x_n \\ x_1', \ldots, x_n' \end{pmatrix}. \tag{C.10}$$

In the thermodynamic limit with no external potential, $U(z) \equiv 0$, the resolvent kernel θ_T is given by

$$\lim_{L\to\infty} \theta_T(x, y) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}k \frac{\mathrm{e}^{\mathrm{i}k(x-y)}}{1 + \mathrm{e}^{(k^2-h)/T}}. \tag{C.11}$$

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